Lecture 16 - Fluids

A Puzzle...

You often find when you drain a bathtub (or sink) that the water will swirl quickly, forming a small whirlpool around the drain. Why does this whirlpool form?

Solution

In a bathtub, even if the water appears to be stationary, there are slight circulating motions going on from when the tub was filled up (these currents can take hours to dissipate away). Because the drain opening is much smaller than the bathtub, as the water gets near the small drain, conservation of angular momentum implies that the water's rotational motion $\dot{\theta}$ must increase. □

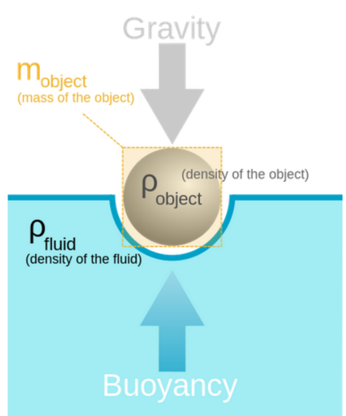
Fluids

Theory

One of the most important forces exerted by a fluid is *buoyancy*. The net upward buoyancy force is equal to the magnitude of the weight of fluid displaced by the body

$$F_b = m_{\text{fluid displaced}}\, g \tag{1}$$

where $m_{\text{fluid displaced}} = \rho_{\text{fluid}}\, V_{\text{object submerged}}$ is the mass of the fluid that is displaced by the object (in the diagram, this equals the volume of the sphere that is submerged).



Often times, it is easier to consider the densities of objects rather than their masses. For example, consider an object with density ρ_{obj} and volume V_{obj} gently placed on top of an infinitely large tub of fluid with density ρ_{fluid}. Will the object float or sink?

Case 1 $\rho_{\text{obj}} > \rho_{\text{fluid}}$

If the object is more dense than the fluid, the object will feel a gravitational force

$$m_{\text{obj}}\, g = \rho_{\text{obj}}\, V_{\text{obj}}\, g \tag{2}$$

downward and a buoyancy force

$$m_{\text{fluid displaced}}\, g = \rho_{\text{fluid}}\, V_{\text{fluid displaced}}\, g \tag{3}$$

upwards. Since $\rho_{\text{obj}} > \rho_{\text{fluid}}$, the volume of displaced water $V_{\text{fluid displaced}}$ (which begins at 0 when the object just comes into contact with the top of the fluid) will increase until the entire object is submerged in the fluid. At this point, $V_{\text{fluid displaced}} = V_{\text{obj}}$ and the net upwards force on the object will be

$$F_{\text{up}} = \rho_{\text{fluid}}\, V_{\text{obj}}\, g - \rho_{\text{obj}}\, V_{\text{obj}}\, g = -(\rho_{\text{obj}} - \rho_{\text{fluid}})\, V_{\text{obj}}\, g \tag{4}$$

Because $\rho_{\text{obj}} > \rho_{\text{fluid}}$, the object will continue to sink down, but instead of doing so with acceleration g it will have a smaller acceleration $\frac{(\rho_{\text{obj}} - \rho_{\text{fluid}})\, V_{\text{obj}}}{m}\, g$ due to the fluid's buoyancy force.

Case 2 $\rho_{\text{fluid}} > \rho_{\text{obj}}$

If the fluid is more dense than the object, and the object is pushed completely into the fluid, then $V_{\text{fluid displaced}} = V_{\text{obj}}$ and, as found above, the upwards force on the object will be

$$F_{\text{up}} = (\rho_{\text{fluid}} - \rho_{\text{obj}})\, V_{\text{obj}}\, g \tag{5}$$

However, in this case, this force is positive, implying that the object will float upwards and breach the surface. It will come into equilibrium when it is partially floating in the fluid (as seen in the image above with the floating sphere) when the buoyancy force and gravitational forces are equal,

$$\rho_{\text{obj}}\, V_{\text{obj}}\, g = \rho_{\text{fluid}}\, V_{\text{fluid displaced}}\, g \tag{6}$$

at which point

$$\rho_{\text{obj}}\, V_{\text{obj}} = \rho_{\text{fluid}}\, V_{\text{fluid displaced}} \tag{7}$$

Advanced Section: Combinations of Liquids

Daily Experience

In our daily lives, we have often experienced a wide variety of fluid mechanics situations (although we generally never categorize them as such). Let's pull on our intuition to answer some questions.

Example

What has a larger density: water or ice?

Solution

Because we know that ice floats in water (for example, when you order water with ice the ice cubes float at the top of the cup). Therefore $\rho_{\text{ice}} < \rho_{\text{water}}$. Indeed, at 0° we have $\rho_{\text{ice}} = 917\ \frac{\text{kg}}{m^3}$ and $\rho_{\text{water}} = 1000\ \frac{\text{kg}}{m^3}$.

Example

What is the density of a human body?

Solution

Many people have experienced that if they dive into a pool and release nearly all of their breadth, they sink. If they take a deep lung full of air and then dive into a pool, they slowly float up. This tells us that $\rho_{\text{human}} \approx \rho_{\text{water}} = 1000\ \frac{\text{kg}}{m^3}$.

Example

What causes humans to float when they inhale a large breath before diving?

Solution

Inhaling air expands your lungs, and $\rho_{\text{air}} < \rho_{\text{water}}$ (since bubbles in water always float upwards), so the buoyancy force on this little pocket of air in your lungs will be larger than its gravitational force. This force is often enough to make the difference between you sinking or floating.

Example

When you drop an anchor from a boat into a lake, does the water level rise, lower, or stay the same?

Solution

An anchor is always made out of a very dense material (normally metal) that is much more dense than water (for example, $\rho_{\text{water}} = 1000\ \frac{\text{kg}}{m^3} < \rho_{\text{steel}} = 7800\ \frac{\text{kg}}{m^3}$). When the anchor is on your boat, the boat must sink down to displace V_{on} of water where

$$\rho_{\text{water}}\, V_{\text{on}}\, g = \rho_{\text{anchor}}\, V_{\text{on}}\, g \tag{8}$$

or equivalently

$$V_{\text{on}} = \frac{\rho_{\text{anchor}}}{\rho_{\text{water}}} V_{\text{anchor}} \tag{9}$$

When the anchor is thrown into the water, then it will completely submerge into the water (because $\rho_{\text{anchor}} > \rho_{\text{water}}$) and it will displace

$$V_{\text{off}} = V_{\text{anchor}} \tag{10}$$

Since $\frac{\rho_{\text{anchor}}}{\rho_{\text{water}}} > 1$, $V_{\text{on}} > V_{\text{off}}$ so that the water level will lower when the anchor is thrown into the water. □

Fluid Flow

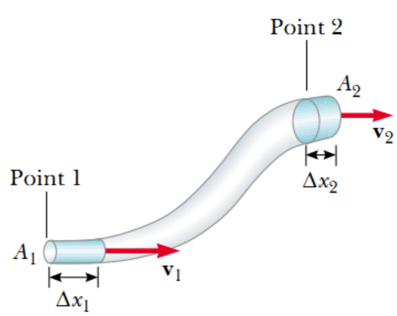
Real fluids are very complicated. To simplify things greatly, we will make four important approximations (none of which are strictly true):

1. Fluids are non-viscous - For example, we know that it is tougher to move a spoon through honey that through water, but we ignore this effect.
2. Fluid flow is steady - The velocity of the fluid at each point remains constant over time.
3. Fluids are incompressible - The density of a fluid is constant over time.
4. Fluid flow is not rotational - A vortex cannot form in the fluid. Eddies and whirlpools don't exist.

With these four assumption, we can make great theoretical strides in understanding fluids; assuming that the deviation from these approximations is not too large, these insights can help us understand how real fluids work.

For example, the incompressibility of fluids and the fact that fluid flow is steady implies that when you restrict the flow from a garden hose it will increase the speed of the flow. More precisely, if a fluid flows at velocity v_1 in a tube with cross-sectional area A_1, and if this cross-sectional area gradually changes to A_2, then the fluid flow at this point must satisfy

$$v_1 A_1 = v_2 A_2 \tag{11}$$

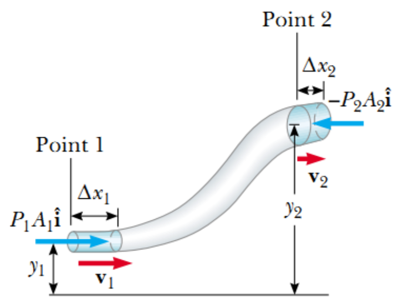


The *Flow* tab in this PHeT Simulation demonstrates the relation between the speed and cross sectional area of an arbitrary pipe.

Complementary Section: Bernoulli’s Principle

Furthermore, we analyze the energy of a fluid moving through a pipe. Before doing so, let’s do a quick recap on pressure. Pressure P equals force over area. Pressure has the named unit of a Pascal, $1\text{ Pa} \equiv 1 \frac{\text{kg}}{m \cdot s^2}$. Atmospheric pressure is denoted by $P_0 = 1\text{ atm} = 10^5\text{ Pa}$.

Now, let us consider the flow of a fluid based on work and energy. Consider a segment of the fluid from Point 1 (the left end of the left shaded cylinder) to Point 2 (the left end of the right shaded cylinder).



In time Δt, this fluid will flow so that the segment spanning Δx_1 across the left cylinder will now span Δx_2 across the right cylinder.

The force $P_1 A_1$ acts along the direction of fluid flow and hence does positive work $P_1 A_1 \Delta x_1$ on the fluid. The force $P_2 A_2$ acts in the opposite direction of fluid flow and hence does negative work $-P_2 A_2 \Delta x_2$ on the fluid. The contributions from all points in between the cylinders Δx_1 and Δx_2 cancel out, since they do negative work when the fluid flows into them and positive work when the fluid flows past them.

As we found above, $A_1 \Delta x_1 = A_2 \Delta x_2$, so we can define this volume as $V \equiv A_1 \Delta x_1 = A_2 \Delta x_2$. Thus the work done on this segment of fluid equals

$$W = (P_1 - P_2) V \tag{12}$$

Now let’s look at the kinetic energy of the fluid during the time span Δt. Assuming steady flow, before the time interval the fluid at Point 1 travels at velocity v_1, whereas after the time interval the fluid at Point 2 travels at v_2. Assuming a constant fluid flow, the unshaded portion of the tube shown above keeps its same velocity. Therefore, the change in kinetic energy equals

$$\Delta\text{KE} = \frac{1}{2} (\rho_{\text{fluid}} V) v_2^2 - \frac{1}{2} (\rho_{\text{fluid}} V) v_1^2 \tag{13}$$

where $\rho_{\text{fluid}} V$ is the mass of the fluid in the shaded tubes at Point 1 and Point 2.

Lastly, we calculate the change in potential energy. Since fluid at Point 1 has been pushed through the unshaded portion of the tube to Point 2, the difference in potential energy after time Δt equals

$$\Delta\text{PE} = (\rho_{\text{fluid}} V) g y_2 - (\rho_{\text{fluid}} V) g y_1 \tag{14}$$

Since the total work done on a system equals the change in energy of the system, we have

$$W = \Delta\text{KE} + \Delta\text{PE} \tag{15}$$

$$(P_1 - P_2) V = \frac{1}{2} (\rho_{\text{fluid}} V) v_2^2 - \frac{1}{2} (\rho_{\text{fluid}} V) v_1^2 + (\rho_{\text{fluid}} V) g y_2 - (\rho_{\text{fluid}} V) g y_1 \tag{16}$$

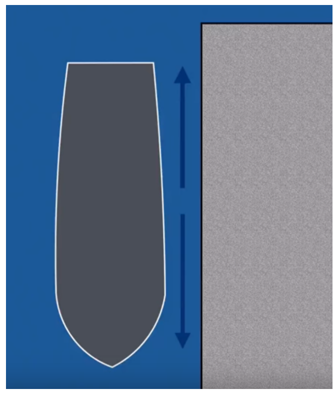
$$P_1 + \frac{1}{2} \rho_{\text{fluid}} v_1^2 + \rho_{\text{fluid}} g y_1 = P_2 + \frac{1}{2} \rho_{\text{fluid}} v_2^2 + \rho_{\text{fluid}} g y_2 \tag{17}$$

In other words,

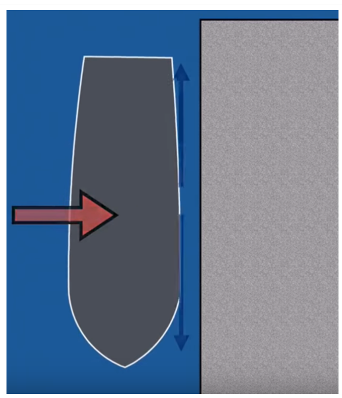
$$P + \frac{1}{2} \rho_{\text{fluid}} v^2 + \rho_{\text{fluid}} g y = \text{constant} \tag{18}$$

Bernoulli’s Principle helps explain many phenomena. For example, boats always dock at piers built upon wooden stakes with water underneath, as shown below. What would happen to a boat trying to dock at a pier built on concrete a cement structure?

As the boat approached the pier, it would have to displace the water between itself and the pier, moving it aside.



This motion would cause the pressure of the water between the boat and the pier to drop, and the difference in pressure between the water to the left of the boat and to the right of the boat would cause a rightwards force on the boat that would slam it into the pier.

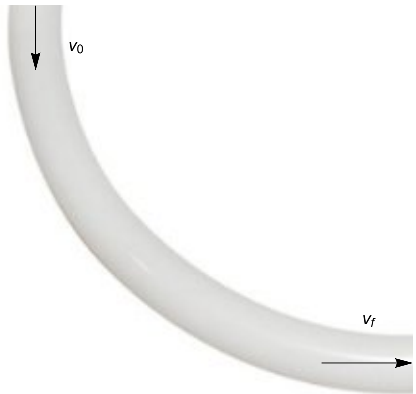


By building a pier on top of wooden stakes that permit water flow, the displaced water simply slides under the pier, preventing the difference in water pressure from building. □

Fluid Problems

<u>Example</u>

Fluid flows in a hose of uniform area A. Fluid comes in at velocity v_0 heading straight down at the top of the pipe and exits with velocity v_f. What is v_f, and how do you justify the change in potential energy?



Solution

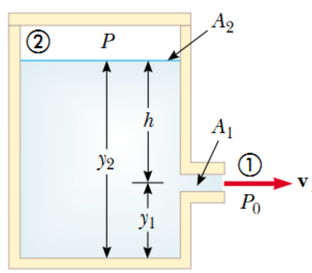
We don't need Bernoulli's equation, since $v_0 A = v_f A$ implies that $v_0 = v_f$. Why does the fluid not speed up due to gravitational potential energy? Here, we can use Bernoulli's equation (assuming a constant velocity) to obtain

$$P + \rho_{\text{fluid}}\, g\, y = \text{constant} \tag{19}$$

so that the gravitational potential energy is responsible for creating a larger pressure for fluid lower down. Note that this result is independent of the shape of the tube (i.e. it could be a larger curve, smaller curve, or simply a straight line), provided that the cross-sectional area of the tube is constant throughout. □

Example

An enclosed tank containing a liquid of density ρ has a hole in its side at a distance y_1 from the tank's bottom. The hole is open to the atmosphere, and its diameter is much smaller than the diameter of the tank ($A_1 \ll A_2$). The air above the liquid is maintained at a pressure P. Determine the speed of the liquid as it leaves the hole when the liquid's level is a distance h above the hole.



Solution

Because $A_1 \ll A_2$, the liquid is approximately at rest at the top of the tank, where the pressure is P. The pressure at the hole must be atmospheric pressure (denoted by P_0). Therefore, applying Bernoulli's equation to Points 1 and 2,

$$P_0 + \frac{1}{2} \rho\, v_1^2 + \rho\, g\, y_1 = P + \rho\, g\, y_2 \tag{20}$$

$$v_1 = \left(\frac{2\,(P - P_0)}{\rho} + 2\, g\, h \right)^{1/2} \tag{21}$$

In the limit $\frac{P-P_0}{\rho} \gg g\, h$, then $v_1 \approx \left(\frac{2\,(P-P_0)}{\rho} \right)^{1/2}$ and the speed at the hole is determined by the pressure in the tank. In the other limit when $\frac{P-P_0}{\rho} \ll g\, h$ (for example, if the top of the tank was open to the atmosphere so that $P = P_0$), then $v_1 \approx (2\, g\, h)^{1/2}$ which is identical to the speed of an object dropped from a height h.

Lets consider further this second limit where $P = P_0$. If we wanted to maximize the horizontal range that water coming out from the hole could travel, what value of y_1 should we choose for the hole?

Remembering our 2D projectile motion, the y-component of a chunk of water coming out of the hole would satisfy

$$y[t] = y_1 - \frac{1}{2} g t^2 \tag{22}$$

We can set $y[t] = 0$ to obtain $t = \left(\frac{2 y_1}{g}\right)^{1/2}$. The horizontal distance traveled equals

$$x[t] = v_1 t \tag{23}$$

which using $v_1 = (2 g h)^{1/2}$ at time $t = \left(\frac{2 y_1}{g}\right)^{1/2}$ yields the horizontal distance

$$\begin{aligned} d &= (2 g h)^{1/2} \left(\frac{2 y_1}{g}\right)^{1/2} \\ &= 2 (h y_1)^{1/2} \\ &= 2 (\{y_2 - y_1\} y_1)^{1/2} \end{aligned} \tag{24}$$

We can maximize this distance by taking the derivative with respect to y_1 and setting it equal to 0,

$$\frac{d}{dy_1} d = (\{y_2 - y_1\} y_1)^{-1/2} (y_2 - 2 y_1) = 0 \tag{25}$$

which yields

$$y_1 = \frac{y_2}{2} \tag{26}$$

That's a very clean result! You can play with an interactive animation in the *Water Tower* tab of this PHeT Simulation. □

Example

When a baseball pitcher throws a ball in the air with no spin, the ball flies straight, with its path dictated solely by gravity. When happens if a ball is thrown with top-spin, so that the top of the ball rolls forward in the direction of the throw?

Solution

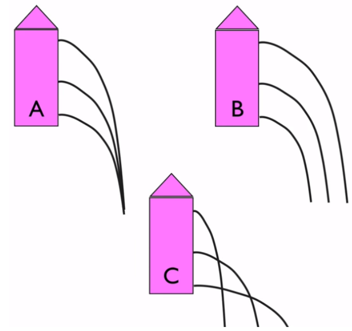
These types of curve balls are commonly used in baseball. If the ball is thrown with no spin, then the speed of air going around both its top and bottom will be the same, as shown below.

If the ball is throw with top-spin, the surface of the ball will drag the air molecules with its motion, thereby reducing the air flow around its top and increasing the air flow around its bottom.

This difference in air pressure will result in a net downwards force on the ball, causing it to fall unexpectedly. Similarly, throwing a ball with bottom-spin will cause it to rise in the air. □

Example

You simultaneously poke three holes at the top, middle, and bottom of a milk carton (below, left). Which of the following stream patterns (below, right) represent the flow of milk out of the carton?



Solution

The lowest hole has the greatest pressure, and hence will shoot milk out the furthest. Thus, C represents the correct streams of milk. □

Some Fun Experiments

- Balloons in Cars: Which way does a balloon lean when a car accelerates?
- Rotating Candles in a Dome: Which way will a flame point inside of a spinning dome? (Explanation found later in the video)
- Balloon Bench: What is the minimum number of balloons you could sit on without popping them?
- Potato vs Straw: Is the pen mightier than the sword? Is the humble straw strong enough to poke a potato straight through?
- Vortices: What crazy behavior can happen when you acknowledge that fluid flow can be rotational?
- Fun Tricks with Water: A little science, a little magic, a little mischief...what more can you want?